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Positive Continuous Linear Functionals on Riesz Spaces and Applications to Minimax Theorems

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1. INTRODUCTION

Throughout this paper, E will denote an order-complete topological Riesz space, with cone C having nonempty interior. This paper is based upon the rather simple property (Lemma 3.1) that the "weak" equality—$\varphi(x)=0$ for every positive continuous linear functional φ on E—is equivalent to the equality $x=0$. According to this result, the key idea of the paper will be deducing minimax equalities for functions taking values in E, from weak minimax equalities (Theorem 3.4) and ultimately via classical minimax theorems for real valued functions.

Consequently a large variety of Riesz space versions of well-known minimax results can be stated under standard conditions. However, it should be noted that in our new setting least upper bounds and greatest lower bounds shall be required to be adherence points explicitly (see conditions (3.4.2) and (3.4.4) below).

For the definitions concerning topological Riesz spaces and the theory of positive continuous linear functionals, we refer to the book [12].

2. LOWER SEMICONTINUOUS FUNCTIONS WITH VALUES IN A RIESZ SPACE

Let X be a topological space.

2.1. DEFINITION. We will say that a function $f\colon X \to E$ is *lower semicontinuous* (l.s.c.) *at* x_0 if for every $\varepsilon \in \mathring{C}$ there exists a neighbourhood U of x_0 such that

$$f(x)-f(x_0)+\varepsilon \in \mathring{C}$$

for all $x \in U$, where $\mathring{C}$ is the interior of C.

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We will say that f is *lower semicontinuous* (l.s.c.) *on* X if f is l.s.c. at x for every $x \in X$.

2.2. PROPOSITION. *If* $f\colon X \to E$ *is continuous, then it is l.s.c.*

Proof. It is enough to recall that for every $\varepsilon \in \mathring{C}$ the order interval $[-\varepsilon, \varepsilon]$ is a neighbourhood of the origin.

2.3. PROPOSITION. *If* $f\colon X \to E$ *is l.s.c. then the level sets* $L_\alpha = \{x \in X\colon f(x) \leqslant \alpha\}$, $\alpha \in E$, *are closed.*

Proof. Fix $\alpha \in E$ and let x_0 be an adherence point for L_α; then there exists a net $(x_i)_{i \in I}$ in L_α such that x_i converges to x_0. By lower semicontinuity, for every $\varepsilon \in \mathring{C}$ there exists an index $\bar{\imath} = \bar{\imath}(\varepsilon)$ such that for every i following $\bar{\imath}$, $f(x_i) - f(x_0) + \varepsilon \in \mathring{C}$. As $x_i \in L_\alpha$, $f(x_i) \leqslant \alpha$ and hence $f(x_0) - \alpha \leqslant \varepsilon$. By the arbitrariness of $\varepsilon \in \mathring{C}$ and since $\inf \mathring{C} = 0$, it follows that $x_0 \in L_\alpha$.

2.4. PROPOSITION. *If* X *is compact and* $f\colon X \to E$ *is l.s.c. then* $\inf\{f(x), x \in X\}$ *exists.*

Proof. By lower semicontinuity, for every fixed $\varepsilon \in \mathring{C}$ and $x_0 \in X$, an open neighbourhood U_{x0} exists such that $f(x) - f(x_0) + \varepsilon \in \mathring{C}$ for $x \in U_{x0}$. The family $\{U_{x0}, x_0 \in X\}$ being an open covering of the compact set X, there exists $\{x_1, ..., x_n\} \subset X$ such that $\bigcup_{j=1}^{n} U_{x_j} = X$. Therefore, setting $i = \inf\{f(x_j), j = 1, ..., n\}$, we find $f(x) \geqslant i - \varepsilon$ for every $x \in X$.

2.5. EXAMPLE. Let $X = \{0\} \cup \{1/n, n \in \mathbb{N}\} \subset \mathbb{R}$ be endowed with the usual topology. Let $E = \mathbb{R}^2$ with the order induced by the cone $C = \{(x, y)\colon x \geqslant 0, y \geqslant 0\}$ and with the usual Euclidean topology of the plane. Consider the function $f\colon X \to E$ defined by

$$f(x) = \begin{cases} (0, 0) & \text{if } x = 0 \\ (-n, n) & \text{if } x = 1/n, n \in \mathbb{N}. \end{cases}$$

One can easily be convinced that the level sets are either empty or finite. On the other hand, by Proposition 2.4, f cannot be l.s.c. as the domain is compact while the range is unbounded. This shows that the converse of Proposition 2.3 is not true in general.

2.6. EXAMPLE. Let $X = [0, 1]$ be endowed with the usual topology and let E be as in the example above. Define $f\colon X \to E$ by $f(x) = (x, 1 - x)$, $x \in X$. The function f is continuous and X is compact, but $\inf\{f(x), x \in X\} = (0, 0)$ is not a minimum. This shows that the l.u.b. in Proposition 2.4 is not a minimum in general, even if f is continuous.

2.7. PROPOSITION. *If $f\colon X \to E$ is l.s.c., then $\varphi \circ f$ is l.s.c. for every $\varphi \in C'$, where C' is the cone of positive continuous linear functionals on E.*

Proof. By continuity of φ at 0, for every fixed real number $t > 0$ there exists a neighbourhood U of 0 such that $\varphi(U) \subset]-t, t[$. Observe that $\mathring{C} \cap U$ is nonempty as $0 \in C = \overline{\mathring{C}}$; then for every $\varepsilon \in \mathring{C} \cap U$ $\varphi(\varepsilon) < t$. Corresponding to $\varepsilon \in \mathring{C} \cap U$, by lower semicontinuity, a neighbourhood V of x_0 can be found such that $f(x) - f(x_0) \in \mathring{C}$ for every $x \in V$. Hence, by linearity and positivity of φ it follows that $\varphi(f(x)) - \varphi(f(x_0)) + t > \varphi(f(x) - f(x_0) + \varepsilon) > 0$, which concludes the proof.

3. Positive Continuous Linear Functionals and a General Minimax Theorem

3.1. LEMMA. *It is $\bigcap_{\varphi \in C'} H(\varphi, 0) = \{0\}$, where $H(\varphi, 0)$ represents the hyperplane $H(\varphi, 0) = \{z \in E\colon \varphi(z) = 0\}$.*

Proof. Obviously $0 \in \bigcap_{\varphi \in C'} H(\varphi, 0)$; to prove that $\bigcap_{\varphi \in C'} H(\varphi, 0) \subset \{0\}$, we will prove that for every $y_0 \in E \setminus \{0\}$, there exists $\varphi \in C'$ such that $\varphi(y_0) \neq 0$. Indeed, as $C \cap -C = \{0\}$, then either $y_0 \notin C$ or $y_0 \notin -C$. Let us suppose that $y_0 \notin C$. Then, by the Archimedean property, there exists $\varepsilon \in \mathring{C}$ such that $y_0 \notin C - \varepsilon$. Since $C - \varepsilon$ is closed, there is a neighbourhood U of y_0 such that $U \cap (C - \varepsilon) = \varnothing$. Without loss of generality we assume U to be open and convex. Therefore we can apply the strong separation theorem [9, Theorem II.9.1] to the convex open sets U and $\mathring{C} - \varepsilon$, namely, by Theorem I.4.2 of [9], there exist $\varphi \in E'$ and $t \in \mathbb{R}$ such that $\varphi(y) > t$ for $y \in U$, and $\varphi(y) < t$ for $y \in \mathring{C} - \varepsilon$. Thus $0 = \varphi(0) < t < \varphi(y_0)$, and, since $0 \in \mathring{C} - \varepsilon$, an interval $[-e, e]$ exists such that $e \in \mathring{C}$ and $\varphi(y) < t$ for every $y \in [-e, e]$. Therefore, by virtue of Proposition 2.8 of [12], $\varphi \in C' - C'$, i.e., there exist $\varphi_i \in C'$, $i = 1, 2$, such that $\varphi = \varphi_1 - \varphi_2$. As $\varphi(y_0) \neq 0$, at least one of the numbers $\varphi_i(y_0) \neq 0$, $i = 1, 2$ must be different from zero. This completes the proof.

3.2. LEMMA. *Given a lower bounded function $f\colon X \to E$, where X is an arbitrary set, such that the following property holds*

$$\inf_{x \in X} f(x) \in \overline{f(x)}, \text{ where } \overline{f(X)} \text{ is the adherence of } f(X), \tag{3.2.1}$$

then, for every $\varphi \in C'$, we have

$$\varphi(\inf_X f(x)) = \inf_X \varphi(f(x)).$$

Proof. As $\inf_X f(x) \leqslant f(\bar{x})$ for every $\bar{x} \in X$, by positivity of φ it follows that $\varphi(\inf_X f(x)) \leqslant \varphi(f(\bar{x}))$ for every $\bar{x} \in X$ and hence $\inf_X \varphi(f(x))$ exists in $\mathbb{R}$. Moreover,

$$\varphi(\inf_X f(x)) \leqslant \inf_X \varphi(f(x)). \tag{3.2.2}$$

From hypothesis (3.2.1) it follows that for every fixed $\varepsilon \in \mathring{C}$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that $\inf_X f(x) \leqslant f(x_n) \leqslant \inf_X f(x) + \varepsilon/n$ for every $n \in \mathbb{N}$. Therefore $\varphi(f(x_n)) \leqslant \varphi(\inf_X f(x)) + \varphi(\varepsilon)/n$, and hence, by (3.2.2), for every $n \in \mathbb{N}$,

$$\varphi(\inf_X f(x)) \leqslant \inf_X \varphi(f(x)) \leqslant \varphi(f(x_n)) \leqslant \varphi(\inf_X f(x)) + \frac{\varphi(\varepsilon)}{n};$$

i.e.,

$$0 \leqslant \inf_X \varphi(f(x)) - \varphi\,(\inf_X f(x)) \leqslant \frac{\varphi(\varepsilon)}{n}. \tag{3.2.3}$$

By the arbitrariness of $n \in \mathbb{N}$, from (3.2.3) we obtain the conclusion.

3.3. COROLLARY. *Let X and Y be two arbitrary sets and let $f \colon X \times Y \to E$ be a function such that:*

$$\textit{for every } y \in Y\ f(\cdot, y) \textit{ is lower bounded on } X; \tag{3.3.1}$$

$$\textit{for every } y \in Y,\ \inf_X f(x, y) \in \overline{f(X \times \{y\})}; \tag{3.3.2}$$

$$\textit{the function } y \mapsto \inf f(x, y) \textit{ is upper bounded on } Y. \tag{3.3.3}$$

Then, for every $\varphi \in C'$ we have

$$\varphi(\sup_Y \inf_X f(x, y)) \geqslant \sup_Y \inf_X \varphi(f(x, y)).$$

Proof. By the positivity of $\varphi \in C'$, from $\sup_Y \inf_X f(x, y) \geqslant \inf_X f(x, \bar{y})$ for every $\bar{y} \in Y$, it follows that

$$\varphi(\sup_Y \inf_X f(x, y)) \geqslant \varphi(\inf_X f(x, \bar{y})) \qquad \text{for every} \quad \bar{y} \in Y,$$

and, by Lemma 3.2.,

$$\varphi(\sup_Y \inf_X f(x, y)) \geqslant \inf_X \varphi(f(x, \bar{y})) \qquad \text{for every} \quad \bar{y} \in Y$$

and hence the conclusion follows.

3.4. THEOREM (Main result). *Let X and Y be two arbitrary sets and let $f: X \times Y \to E$ be a function satisfying*:

$$\text{for every } y \in Y\ f(\cdot, y) \text{ is lower bounded on } X; \tag{3.4.1}$$

$$\text{for every } y \in Y \inf_X f(x, y) \in \overline{f(X \times \{y\})}; \tag{3.4.2}$$

$$\text{for every } x \in X\ f(x, \cdot) \text{ is upper bounded on } Y; \tag{3.4.3}$$

$$\text{for every } x \in X \sup_Y f(x, y) \in \overline{f(\{x\} \times Y)}. \tag{3.4.4}$$

If for every $\varphi \in C'$ the equality

$$\inf_X \sup_Y \varphi(f(x, y)) = \sup_Y \inf_X \varphi(f(x, y)) \tag{3.4.5}$$

holds, then

$$\inf_X \sup_Y f(x, y) = \sup_Y \inf_X f(x, y).$$

Proof. Observe that our assumptions ensure that both sides of the equation exist. Indeed, from hypothesis (3.4.3), $\sup_Y f(\bar{x}, y)$ exists for every $\bar{x} \in X$, and by the trivial inequality $f(\bar{x}, y) \leqslant \sup_Y f(\bar{x}, y)$, $\bar{x} \in X$, assumption (3.4.1) ensures that $\inf_X \sup_Y f(x, y)$ exists. Moreover, since $\inf_X \sup_Y f(x, y) \geqslant \inf_X f(x, \bar{y})$, $\bar{y} \in Y$, $\sup_Y \inf_X f(x, y)$ also exists. Therefore, by Corollary 3.3 we get $\sup_Y \inf_X \varphi(f(x, y)) \leqslant \varphi(\sup_Y \inf_X f(x, y))$ and by the symmetry of assumptions we also have $\varphi(\inf_X \sup_Y f(x, y)) \leqslant \inf_X \sup_Y \varphi(f(x, y))$. Thus both terms in (3.4.5) exist in $\mathbb{R}$ and, moreover,

$$\varphi(\inf_X \sup_Y f(x, y)) \leqslant \varphi(\sup_Y \inf_X f(x, y)). \tag{3.4.6}$$

As it is always $\sup_Y \inf_X f(x, y) \leqslant \inf_X \sup_Y f(x, y)$, by positivity of $\varphi \in C'$ and by (3.4.6) it follows that

$$\varphi(\inf_X \sup_Y f(x, y)) = \varphi(\sup_Y \inf_X f(x, y)). \tag{3.4.7}$$

Finally, for every $\varphi \in C'$ $[\inf_X \sup_Y f(x, y) - \sup_Y \inf_X f(x, y)] \in H(\varphi, 0)$ and the conclusion is a consequence of Lemma 3.1.

4. A MINIMAX THEOREM WITHOUT LINEAR STRUCTURE

This section is devoted to the transposition to Riesz space-valued functions of a well-known minimax theorem not involving linear structures of the domain.

4.1. THEOREM. *Let X be a compact topological space and Y be a given set. Let $f: X \times Y \to E$ be a function satisfying*:

$$\text{for every } y \in Y \; f(\cdot, y) \text{ is l.s.c. on } X; \tag{4.1.1}$$

$$\text{for every } y \in Y \; \inf_X f(x, y) \in \overline{f(X \times \{y\})}; \tag{4.1.2}$$

$$\text{for every } x \in X \; f(x, \cdot) \text{ is upper bounded on } Y; \tag{4.1.3}$$

$$\text{for every } x \in X \; \sup_Y f(x, y) \in \overline{f(\{x\} \times Y)}; \tag{4.1.4}$$

there exist $s_1, s_2 \in \mathbb{R}^+$ such that, for every $\varepsilon \in \mathring{C}$ and for every $x_1, x_2 \in X$ there exists $x_0 \in X$ with

$$f(x_0, y) \leqslant s_1 f(x_1, y) + s_2 f(x_2, y) + \varepsilon \text{ for every } y \in Y; \tag{4.1.5}$$

there exist $t_1, t_2 \in \mathbb{R}^+$ such that, for every $\varepsilon \in \mathring{C}$ and for every $y_1, y_2 \in Y$ there exists $y_0 \in Y$ with

$$t_1 f(x, y_1) + t_2 f(x, y_2) \leqslant f(x, y_0) + \varepsilon \text{ for every } x \in X. \tag{4.1.6}$$

Then

$$\inf_X \sup_Y f(x, y) = \sup_Y \inf_X f(x, y).$$

Proof. Observe that, by Propositions 2.4 and 2.7, the assumption (3.4.1) in Theorem (3.4) is satisfied and, moreover, $\varphi \circ f$ is l.s.c. on X for every $\varphi \in C'$. Furthermore, by continuity of each $\varphi \in C'$, for every $\delta \in \mathbb{R}^+$ there exists a neighbourhood U of 0 such that $0 < \varphi(\varepsilon) < \delta$ for every $\varepsilon \in \mathring{C} \cap U$. Therefore, for every $\delta \in \mathbb{R}^+$ and for every $x_1, x_2 \in X$, by (4.1.5) we have

$$\varphi(f(x_0, y)) \leqslant s_1 \varphi(f(x_1, y)) + s_2 \varphi(f(x_2, y)) + \delta$$

which is Fuchsteiner and König's convex-likeness [3]. Analogously, by (4.1.6), Fuchsteiner and König's concave-likeness assumption can be obtained. Hence, for every $\varphi \in C'$, Theorem 5.2 in [3] gives

$$\inf_X \sup_Y \varphi(f(x, y)) = \sup_Y \inf_X \varphi(f(x, y))$$

and by Theorem 3.4 the assertion follows.

4.2. *Remark.* Theorem 4.1 is the Riesz space version of a minimax theorem due to Fuchsteiner and König [3]. Similarly, minimax theorems analogous to those of Kneser [7], Ky Fan [2], Sion [10], Wu [13], Brezis, Nirenberg and Stampacchia [1], Terkelsen [11], Ha [5, 6], and Geraghty and Lin [4] can be obtained for the Riesz space setting.

In [3], besides the above-mentioned minimax result for semi-continuous functions, a minimax theorem for one-side bounded functions has been given. Recently, we have deduced in [8] another minimax theorem for one-side bounded functions from a Han–Banach type result.

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